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A note on slow vibrations in a viscous fluid

By W. E. WILLIAMS

Department of Mathematics, Battersea College of Technology, London, S.W.11

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An approximate expression is obtained for the force on an arbitrary body executing slow vibrations in a viscous fluid.

1. Introduction

This note is concerned with the determination of the force on a slowly vibrating body in a viscous fluid. The problem is a generalization of one examined by Kanwal (1964), who considered the particular case of an axisymmetric body vibrating along its axis of symmetry. Kanwal asserts that his expressions for the force are correct for arbitrary bodies; it will, however, be shown that this assertion is not generally valid.

The method employed is an integral-equation one closely allied to that used by the author (Williams 1966) in treating boundary effects in steady Stokes flow.

2. Detailed formulation and solution of the problem

It will be assumed that the body oscillates about some mean position with velocity $Ue^{i\omega t}\mathbf{i}$, where $\mathbf{i}$ is a unit vector, and throughout the work dimensionless space co-ordinates based on a typical dimension a of the body S will be used. The time-dependent Stokes flow equations may be written in the form (the exponential time factor will be suppressed through the work)

$$iM^2\mathbf{q} = -\operatorname{grad} p + \nabla^2\mathbf{q}, \quad \operatorname{div}\mathbf{q} = 0, \tag{1}$$

where

$$\mathbf{q} = \mathbf{q}'/U, \quad p = ap'/\mu U, \quad M^2 = a^2\omega\rho/\mu,$$

ρ is the density, μ the viscosity and $\mathbf{q}'$ and p' denote the velocity and pressure expressed in physical units.

The first step is to obtain an appropriate integral representation for $\mathbf{q}$; such a representation may be obtained by an immediate extension of the known results for steady Stokes flow (Williams 1966). It follows immediately that the value of $\mathbf{q}$ at a point P in the region exterior to S is given by

$$\mathbf{q} = -\int_S \left\{\mathbf{f}.\mathbf{T} - \mathbf{q}.\left(\frac{\partial \mathbf{T}}{\partial n} - \mathbf{pn}\right)\right\} dS, \tag{2}$$

where $\mathbf{f} = \partial\mathbf{q}/\partial n - p\mathbf{n}$ and $\mathbf{n}$ denotes the unit vector along the outward normal to S. The tensor $\mathbf{T}$ and the vector $\mathbf{p}$ are solutions of the equations

$$iM^2T = -\operatorname{grad}\mathbf{p} + \nabla^2\mathbf{T} + \mathbf{U}\delta(\mathbf{r}-\mathbf{r}_0), \quad \operatorname{div}\mathbf{T} = 0, \tag{3}$$

where $\mathbf{U}$ is the unit tensor, $\mathbf{r}_0$ denotes an arbitrary point on S and $\mathbf{r}$ denotes the position vector of P. The tensor $\mathbf{T}$ is also required to tend to zero as $|\mathbf{r}-\mathbf{r}_0| \to \infty$.

The solution of equations (3) is clearly

$$\left.\begin{aligned} \mathbf{T} &= \mathbf{U}\nabla^2\phi - \operatorname{grad}\operatorname{grad}\phi, \quad \mathbf{p} = -\operatorname{grad}(\nabla^2 - iM^2)\,\phi, \\ \text{where} \quad (\nabla^2 - iM^2)\,\nabla^2\phi &= -\delta(\mathbf{r}-\mathbf{r}_0), \end{aligned}\right\} \tag{4}$$

and the appropriate solution of equation (4) is

$$\phi = (1 - \exp\{-k|\mathbf{r}-\mathbf{r}_0|\})/(4\pi k^2|\mathbf{r}-\mathbf{r}_0|),$$

where $k = (1+i)\,M/\sqrt{2}$. For small M

$$\mathbf{T} = \mathbf{T}_0 - k\mathbf{U}/6\pi + O(M^2), \tag{5}$$

where $\quad \mathbf{T}_0 = (\mathbf{U}\nabla^2|\mathbf{r}-\mathbf{r}_0| - \operatorname{grad}\operatorname{grad}|\mathbf{r}-\mathbf{r}_0|)/8\pi.$

From the definition of the stress tensor it can be shown that the force $\mathbf{F}$ on the body is given by

$$\mathbf{F} = \int \mathbf{f}\, dS. \tag{6}$$

The integral equation governing the motion is obtained by taking P in equation (2) to be on S and setting $\mathbf{q} = \mathbf{i}$ on S. As $\mathbf{q}$ is constant on S it may be taken outside the integral in equation (2) and it may be shown from equations (3) and Green's theorem that

$$\int_S \left(\frac{\partial \mathbf{T}}{\partial n} - \mathbf{pn}\right) dS = iM^2 \int_V \mathbf{T}\, dV, \tag{7}$$

where V denotes the interior of S. Equations (2), (5), (6) and (7) thus give, for P on S,

$$\mathbf{i} - k\mathbf{F}/6\pi = -\int \mathbf{T}_0 . \mathbf{f}\, dS + O(M^2). \tag{8}$$

When $k = 0$ the integral equation (8) reduces to that for steady Stokes flow and thus if the Stokes resistance tensor $\mathbf{\Phi}$, defined to be such that the force exerted on a body moving with uniform velocity $\mathbf{u}$ is $\mathbf{\Phi}.\mathbf{u}$, is introduced it follows that the force on the body is

$$F = \mathbf{\Phi}.\{\mathbf{i} - (1+i)\,M(\mathbf{\Phi}.\mathbf{i})/6\pi\sqrt{2}\} + O(M^2). \tag{9}$$

For a body moving parallel to one of its principal axes of resistance (i.e. those directions such that $\mathbf{\Phi}.\mathbf{u}$ is of the form $-D\mathbf{u}$) equation (9) becomes

$$\mathbf{F} = -D\{1 + (1+i)\,MD/6\pi\sqrt{2}\}\,\mathbf{i}. \tag{10}$$

On reverting to physical units equation (10) reduces to the form derived by Kanwal (1964). Kanwal's general result is that the drag on the body is given by the magnitude of the right-hand side of (10) where D is interpreted as the drag in Stokes flow. Clearly this is not true in general for arbitrary motion of a general body, as can be seen by considering the particular case of a circular disk moving at an angle α to its axis of symmetry. Equation (9) gives

$$\mathbf{F} = -\tfrac{16}{3}\cos\alpha\,\mathbf{i}_2(1 + 8(1+i)/9\pi\sqrt{2}) - \tfrac{32}{3}\sin\alpha\,\mathbf{i}_1(1 + 16(1+i)\,M/9\pi\sqrt{2}),$$

where $\mathbf{i}_2$ is a unit vector along the axis of symmetry and $\mathbf{i}_1$ is a unit vector in the plane of the disk and of the direction of motion.

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